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THE FREQUENCY EQUATION THEOREMS OF SMALL OSCILLATIONS OF A HYBRID SYSTEM CONTAINING COUPLED DISCRETE AND CONTINUOUS SUBSYSTEMS

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Katica (Stevanović) Hedrih

Faculty of Mechanical Engineering University of Niš, 18000 Niš, Vojvode Tankosića 3/22,
e-mail: katica@masfak.ni.ac.yu / khedrih@eunet.yu,Telefax: +381 18 241-663

Abstract. *In this paper, by using examples of hybrid systems of a dynamically coupled discrete subsystem of rigid bodies and continuous subsystem, the method for obtaining frequency equations of small oscillations is presented. Also, a theorem of small oscillations frequency equations is defined. By using examples, analogy between frequency equations of some classes of these systems is identified. Special cases of discretization and continualization of coupled subsystems in the light of these sets of proper circular frequencies and frequency equations of small oscillations are analyzed. The set of the frequency equation theorems of small oscillations of the hybrid system containing coupled discrete and continuous subsystems are defined.*

Key words: *Hybrid systems, coupled subsystems, rigid bodies, continuous, elastic body, frequency equation, theorem.*

1. INTRODUCTION

Current research in the theory of discrete dynamical system oscillations is directed to nonlinear phenomena, as well as to nonstationary processes, and also to stochastic-like and chaotic-like processes in purely deterministic dynamical systems and conditions. In the theory of oscillations of continuous systems also nonlinear phenomena and damage and fracture structure of dynamical systems are topics of some premier journals and international scientific meetings and conferences. Pure elastic systems are now not in the focus of researchers.

In accordance with close specializations of researchers we can not find more examples which considered mixed systems consisting of coupled discrete and continuous subsystems (see Refs. [2] and [3]). And not very often are there some analytical results. We can conclude that new computer tools with power computer possibilities directed philosophy of considerations of real systems dynamics by using discretization of continuum as the way and method for solutions of problems, and by using many iterations continualizations

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of solutions. Discretizations and continualizations (see Ref. [18]) in the process of solution and analysis of dynamical processes are opposite directions and good method for proving calculations and conclusions.

In the époque of large numerical experiments over dynamical systems I think that it is very important to make some new classical examples of frequency equations useful for teaching process in the theory of linear vibrations.

In the papers [2] and [3], by using examples of mixed systems of statically coupled discrete subsystem of rigid bodies and continuous subsystem, the method for obtaining of small oscillations frequency equations is presented. Small oscillations frequency equations of coupled deformable body and holonomic conservative systems are obtained. Using numerical experiment connections between own small oscillations, circular frequencies of the mixed system and subsystem of the rigid bodies and deformable body are study and analyzed. Using the MathCAD software graphical presentations of a set of small oscillations circular frequencies of the deformable body with "perturbations" caused on interaction of subsystem small oscillations of rigid bodies were obtained. Through examples, the analogy between frequency equations of some classes of these systems is identified. Special cases of discretization and continualization of coupled subsystems in the light of these sets of proper circular frequencies and frequency equations of small oscillations are analyzed.

These examples are very important as a basis of the hybrid system research (see Refs. from [4] to [18]).

2. INTRODUCTION MODEL OF A HYBRID SYSTEM CONTAINING TWO SUBSYSTEMS AS DYNAMICALLY COUPLED DISCRETE SUBSYSTEM OF RIGID BODIES AND CONTINUOUS SUBSYSTEM

Let us consider two subsystems: one elastic rod, whose axis is straight, as a continuous system solid deformable body with following parameters: $\mathbf{E}$, ρ, ℓ, $\mathbf{A}$, and with two rigid bodies on the ends with weights at the free ends with masses m_p and m_0 (see Figure 1.). This rod and a discrete system with n degree of freedom are coupled by dynamical constraint in the form of rolling element with kinetic parameters m, $\mathbf{J}_c$, R. For example, this discrete subsystem is chain system of the n material particles with masses m_i, $i = 1,2,3,...,n$, translatory movable along line parallel to the rod's axis; these masses are connected by springs with rigidities c_i, $i = 1,2,3,...,n$. We consider connections between longitudinal vibrations of the elastic rod and free oscillations of the chain material particles system. Let us determine the frequency equations of the defined mixed system of the coupled discrete subsystem of rigid bodies and continuous subsystem.

Now, we consider two linear subsystems of discrete material particles with n degrees of freedom and also with $n + 1$ degrees of freedom, and we choose n generalized coordinates x_i, $i = 1,2,3,...,n$, and corresponding matrix $\mathbf{A}$ (and $\widetilde{\mathbf{A}}$) of inertia coefficients and matrix $\mathbf{C}$ (and $\widetilde{\mathbf{C}}$) of quazielastic coefficients:

$$\begin{aligned} \mathbf{A} &= (a_{ij}), \ \ i = 1,2,3,...,n; \quad j = 1,2,3,...,n \\ \mathbf{C} &= (c_{ij}), \ \ i = 1,2,3,...,n; \quad j = 1,2,3,...,n \end{aligned} \tag{1}$$

$$\widetilde{\mathbf{A}} = (\widetilde{a}_{ij}),\ i = 0,1,2,3,...,n;\quad j = 0,1,2,3,...,n$$
$$\widetilde{\mathbf{C}} = (\widetilde{c}_{ij}),\ i = 0,1,2,3,...,n;\quad j = 0,1,2,3,...,n \tag{2}$$

where

$$\widetilde{a}_{00} = \hat{a}_{00},\quad \widetilde{a}_{11} = \hat{a}_{11} + a_{11},\quad \widetilde{a}_{01} = \widetilde{a}_{10} = \hat{a}_{01},\quad \widetilde{a}_{0i} = \widetilde{a}_{i0} = 0\quad \widetilde{a}_{ij} = a_{ij},\quad i,j \neq 0,1$$
$$\widetilde{c}_{00} = 0,\ \widetilde{c}_{11} = c_{11},\ \widetilde{c}_{0i} = \widetilde{c}_{i0} = 0,\ \widetilde{c}_{ij} = c_{ij},\quad i,j \neq 0,1,\ i = 1,2,3,...,n;\quad j = 1,2,3,...,n \tag{3}$$

and with external excitation in the form of the force $\mathbf{F}(t)$ of the subsystem dynamical interaction as a generalized force to the coordinate $u(\ell,t) = x_0(t) = \mathbf{X}(\xi)\mathbf{T}(t)$.

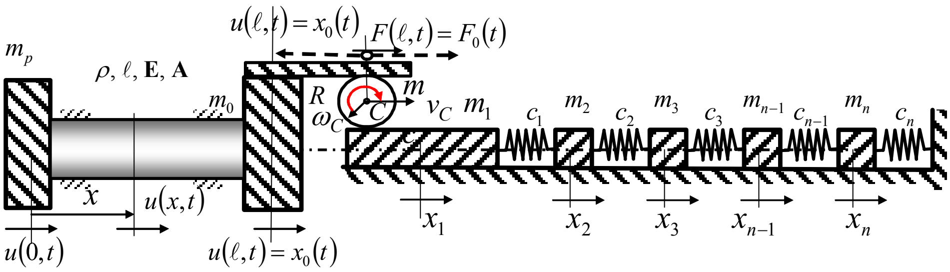

Fig. 1. Small oscillations of the mixed system of coupled discrete and continuous subsystems

2.1. Kinetic energy of the dynamical constraint – coupling element

Angular velocity of the disk rotation, with mass m and mass inertia axial moment $\mathbf{J_C}$, between two masses m_0 and m_1 is: $\omega_C = \frac{\dot{x}_1 - \dot{x}_0}{2R}$, and velocity of the mass center is: $v_C = \frac{\dot{x}_0 + \dot{x}_1}{2}$. Kinetic energy of the coupling nonlinear and linear systems is:

$$\mathbf{E}_{k(1,2)} = \frac{1}{2}[mv_C^2 + \mathbf{J_C}\omega_C^2] = \frac{1}{2}\left[m\left(\frac{\dot{x}_0 + \dot{x}_1}{2}\right)^2 + \mathbf{J_C}\left(\frac{\dot{x}_1 - \dot{x}_0}{2R}\right)^2\right]. \tag{4}$$

or

$$E_{k(1,2)} = \frac{1}{2}\left[\frac{m}{4} + \frac{\mathbf{J_C}}{4R^2}\right]\dot{x}_0^2 + \frac{1}{2}\left[\frac{m}{4} + \frac{\mathbf{J_C}}{4R^2}\right]\dot{x}_1^2 + \frac{1}{2}2\dot{x}_1\dot{x}_0\left[\frac{m}{4} - \frac{\mathbf{J_C}}{4R^2}\right] \tag{5}$$

$$E_{k(1,2)} = \frac{1}{2}(\hat{a}_{00}\dot{x}_0^2 + \hat{a}_{11}\dot{x}_1^2 + 2\dot{x}_1\dot{x}_0\hat{a}_{01}) \tag{6}$$

where

$$\hat{a}_{00} = \frac{m}{4} + \frac{\mathbf{J_C}}{4R^2},\quad \hat{a}_{11} = \frac{m}{4} + \frac{\mathbf{J_C}}{4R^2}\quad \text{and}\quad \hat{a}_{01} = \frac{m}{4} - \frac{\mathbf{J_C}}{4R^2}. \tag{7}$$

Then we have a hybrid system with dynamic, but also linear, constraint between subsystems as dynamic coupled subsystems.

2.2. Differential equation of the longitudinal oscillations of the elastic rod and boundary conditions

In accordance with the notations in the Figure 1 $u(x,t)$ represents longitudinal displacement of the rod's cross section at the distance x measured from the left rod's end in the axis direction at the time t. Partial differential equation of the longitudinal oscillations is:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c_e^2 \frac{\partial^2 u(x,t)}{\partial x^2} \tag{8}$$

where is: $c_e^2 = \frac{\mathbf{E}}{\rho}$.

Solution of (1) is in the following form:

$$u(x,t) = \mathbf{X}(x)\mathbf{T}(t) \tag{9}$$

where:

$$\mathbf{X}(x) = C_1 \cos \lambda x + C_2 \sin \lambda x \tag{10}$$
$$\mathbf{T}(t) = A \cos \omega t + B \sin \omega t$$

Longitudinal oscillations of the beam with multi body chain with changeable numbers of material particles.

Using the boundary conditions of the subsystem of the longitudinal rod's oscillations (see Ref. [1]) as well as the compatibility conditions of displacements and forces as interactions of the coupled subsystems we can write:

$$\left[m_P \frac{\partial^2 u(x,t)}{\partial t^2} \right]\Bigg|_{x=0} = \left[\mathbf{EA} \frac{\partial u(x,t)}{\partial x} \right]\Bigg|_{x=0} \tag{11}$$

$$\left[m_0 \frac{\partial^2 u(x,t)}{\partial t^2} \right]\Bigg|_{x=\ell} = \left[-\mathbf{EA} \frac{\partial u(x,t)}{\partial x} \right]\Bigg|_{x=\ell} + \mathbf{F}(t) \tag{12}$$

where

$$u(\ell,t) = x_0(t) = \mathbf{X}(\xi)\mathbf{T}(t) \tag{13}$$

Taking into account that

$$\{x\} = \{A\}\mathbf{T}(t) \quad \text{and} \quad \ddot{x}_1 = -\omega^2 A_1 \mathbf{T}(t) = -\xi^2 \omega_0^2 A_1 \mathbf{T}(t) \tag{14}$$

the force $\mathbf{F}(t)$ of the subsystem dynamical interaction can be represented in the following form:

$$\mathbf{F}(t) = -[\hat{a}_{00}\ddot{x}_0(t) + \hat{a}_{01}\ddot{x}_1(t)] = \xi^2\omega_0^2[\hat{a}_{00}\mathbf{X}(\xi) + \hat{a}_{01}A_1]\mathbf{T}(t) \tag{15}$$

Let us introduce the following notations:

$$\mu_P = \frac{m_P}{\rho \mathbf{A} \ell} \qquad \mu_o = \frac{m_0}{\rho \mathbf{A} \ell} \qquad \xi = \lambda \ell$$

$$\omega^2 = \lambda^2 \frac{\mathbf{E}}{\rho} = \frac{\xi^2}{\ell^2} \frac{\mathbf{E}}{\rho} = \xi^2 \omega_0^2 \qquad \omega_0^2 = \frac{\mathbf{E}}{\rho \ell^2}$$

$$\frac{d\mathbf{X}(x)}{dx}=\lambda\frac{d\mathbf{X}(\xi)}{d\xi}=\frac{\xi}{\ell}\frac{d\mathbf{X}(\xi)}{d\xi}, \qquad c_e=\frac{\mathbf{EA}}{\ell},$$

$$\hat{\mu}_{00}=\frac{\hat{a}_{00}}{\rho\mathbf{A}\ell}, \quad \hat{\mu}_{01}=\frac{\hat{a}_{01}}{\rho\mathbf{A}\ell}, \quad u_0=\frac{m_0\omega_0^2}{c_0} \tag{16}$$

$$\frac{1}{c_0}\mathbf{C}=\overline{\mathbf{C}} \qquad \frac{1}{m_0}\mathbf{A}=\overline{\mathbf{A}}, \qquad \tilde{\mu}_{01}=\frac{\hat{a}_{01}}{m_0}, \qquad \tilde{\mu}_{11}=\frac{\hat{a}_{11}}{m_0}.$$

And by introducing the proposed solutions (9) into boundary conditions and conditions of the compatibility displacements and forces we can write:

$$\mu_P\xi^2\mathbf{X}(0)+\xi\mathbf{X}'(0)=0 \tag{17}$$

$$\xi^2[\mu_0+\hat{\mu}_{00}]\mathbf{X}(\xi)-\xi\mathbf{X}'(\xi)=\xi^2\hat{\mu}_{01}A_1 \tag{18}$$

Using $\mathbf{X}(\xi)=C_1\cos\xi+C_2\sin\xi$ and the corresponding derivative with respect to the argument ξ: $\mathbf{X}'(\xi)=-C_1\sin\xi+C_2\cos\xi$, from previous equations we can obtain:

$$\mu_P\xi^2C_1+\xi C_2=0 \tag{19}$$

$$\xi C_1[\xi(\mu_0+\hat{\mu}_{00})\cos\xi+\sin\xi]+\xi C_2[\xi(\mu_P+\hat{\mu}_{00})\sin\xi-\cos\xi]=\xi^2\hat{\mu}_{01}A_1$$

where A_1 is unknown amplitude of the first generalized coordinate x_1.

The determinant of the previous system of algebraic equations with respect to the C_1, C_2 is:

$$\Delta(\xi)=\begin{vmatrix}\mu_P\xi^2 & \xi\\ \xi[\xi(\mu_0+\hat{\mu}_{00})\cos\xi+\sin\xi] & \xi[\xi(\mu_0+\hat{\mu}_{00})\sin\xi-\cos n\xi]\end{vmatrix}$$

$$\Delta(\xi)=\xi^2\{\mu_P\xi[\xi(\mu_0+\hat{\mu}_{00})\sin\xi-\cos\xi]-[\xi(\mu_0+\hat{\mu}_{00})\cos\xi+\sin\xi]\} \tag{20}$$

and these coefficients we can express by following expressions:

$$C_1=-\frac{\hat{\mu}_{01}A_1\xi}{\{\mu_P\xi[\xi(\mu_0+\hat{\mu}_{00})\sin\xi-\cos\xi]-[\xi(\mu_0+\hat{\mu}_{00})\cos\xi+\sin\xi]\}} \tag{21}$$

$$C_2=\frac{\mu_P\hat{\mu}_{01}A_1\xi^2}{\{\mu_P\xi[\xi(\mu_0+\hat{\mu}_{00})\sin\xi-\cos\xi]-[\xi(\mu_0+\hat{\mu}_{00})\cos\xi+\sin\xi]\}} \tag{22}$$

2.3. Differential equations of the material particles in the discrete subsystem with boundary conditions

Now, we consider two subsystems of discrete material particles with *n* degrees of freedom and also with $n+1$ degrees of freedom, and we choose *n* generalized coordinates x_i, $i = 1,2,3,...,n$, and the corresponding matrix **A** (and $\tilde{\mathbf{A}}$) of inertia coefficients and matrix **C** (and $\tilde{\mathbf{C}}$) quazielastic coefficients and with external excitation in the form of the force **F**(*t*) of the subsystem dynamical interaction as a generalized force to the coordinate $u(\ell,t)=x_0(t)=\mathbf{X}(\xi)\mathbf{T}(t)$.

System of the differential equations of the discrete subsystem with corresponding boundary conditions in the matrix form is:

$$\widetilde{\mathbf{A}}\{\ddot{\widetilde{x}}\} + \widetilde{\mathbf{C}}\{\widetilde{x}\} = -\mathbf{F}(t)\widetilde{\mathbf{I}}_0\{I\} \tag{23}$$

where is:

$$\{\widetilde{x}\} = \begin{Bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{Bmatrix} \qquad \widetilde{\mathbf{I}}_0 = \begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix} \qquad \{I\} = \begin{Bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{Bmatrix} \tag{24}$$

The dimension of the matrix $\widetilde{\mathbf{I}}_0$ is $n \times n$ and matrix $\{I\}$ $n \times 1$,
Also we can write:

$$\hat{a}_{00}\ddot{x}_0 + \hat{a}_{01}\ddot{x}_1 = -\mathbf{F}(t)$$

$$\mathbf{A}\{\ddot{x}\} + \mathbf{C}\{x\} = -(\hat{a}_{01}\ddot{x}_0 + \hat{a}_{11}\ddot{x}_1)\mathbf{I}_0\{I\} \tag{23*}$$

Where

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{Bmatrix} \qquad \mathbf{I}_0 = \begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix} \tag{24*}$$

The solution of the previous system (23*) is assumed in the following form:

$$\{x\} = \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_{n-1} \\ A_n \end{Bmatrix} \mathbf{T}(t) = \{A\}\mathbf{T}(t) \qquad \{\ddot{x}\} = -\omega^2\{A\}\mathbf{T}(t) = -\xi^2\omega_0^2\{A\}\mathbf{T}(t) \tag{25}$$

2.4. Frequency equation of the coupled longitudinal oscillations of the elastic rod and discrete system of the material particles

Taking into consideration (21)-(22) we can write

$$\begin{aligned} u(\ell,t) &= x_0(t) = \mathbf{X}(\ell)\mathbf{T}(t) \\ u(\ell,t) &= (C_1\cos\xi + C_2\sin\xi)\mathbf{T}(t) = -\frac{\hat{\mu}_{01}A_1\xi^3}{\Delta(\xi)}(\cos\xi - \mu_P\xi\sin\xi)\mathbf{T}(t) \end{aligned} \tag{26}$$

And taking into account that: $\frac{1}{c_0}\mathbf{C} = \overline{\mathbf{C}}$ and $\frac{1}{m_0}\mathbf{A} = \overline{\mathbf{A}}$, form the system of differential equations in matrix form (23) we can obtain the following matrix equation:

$$\mathbf{A}\{\ddot{x}\} + \mathbf{C}\{x\} = -(\hat{a}_{00}\ddot{x}_0 + \hat{a}_{11}\ddot{x}_1)\mathbf{I}_0 \tag{27}$$

Taking into account expressions (26) and (26) and corresponding second derivatives with respect to time we obtain the following matrix equation with respect to unknown amplitudes {A}:

$$\left(\overline{\mathbf{C}} - \xi^2 u_0\left(\overline{\mathbf{A}} + \left[\tilde{\mu}_{11} - \tilde{\mu}_{01}\frac{\hat{\mu}_{01}\xi^3}{\Delta(\xi)}(\cos\xi - \mu_P\xi\sin\xi)\right]\mathbf{I}_0\right)\right)\{A\} = \{0\} \tag{28}$$

Previous matrix equations are algebraic homogeneous equations and for nontrivial solutions it is necessary that determinant of this system is equal to zero. From this condition we can obtain the following characteristic frequency transcendent equation:

$$\left|\overline{\mathbf{C}} - \xi^2 u_0\left(\overline{\mathbf{A}} + \left[\tilde{\mu}_{11} - \tilde{\mu}_{01}\frac{\hat{\mu}_{01}\xi^3}{\Delta(\xi)}(\cos\xi - \mu_P\xi\sin\xi)\right]\mathbf{I}_0\right)\right| = 0 \tag{29}$$

Where

$$\Delta(\xi) = \xi^2\{\mu_P\xi[\xi(\mu_0 + \hat{\mu}_{00})\sin\xi - \cos\xi] - [\xi(\mu_0 + \hat{\mu}_{00})\cos\xi + \sin\xi]\}.$$

This is the main result of this consideration of the mixed system of coupled subsystems free oscillations. We can see that this equation consists of two parts: one part is expression of the frequency equations of the discrete system oscillations, and second of the deformable body frequency equation connected by one member with previous, which is correction of the first inertia matrix coefficient

$$\overline{a}^*_{11} = \overline{a}_{11} + \left[\tilde{\mu}_{11} - \tilde{\mu}_{01}\frac{\hat{\mu}_{01}\xi^3}{\Delta(\xi)}(\cos\xi - \mu_P\xi\sin\xi)\right]. \tag{30}$$

or that

$$a^*_{11} = m_0\overline{a}^*_{11} = a_{11} + \left[\hat{a}_{11} - \hat{a}_{01}\frac{\hat{\mu}_{01}\xi^3}{\Delta(\xi)}(\cos\xi - \mu_P\xi\sin\xi)\right] \tag{30*}$$

From last expression (30) we can conclude that interaction between the subsystems dynamically coupled is expressed by the coefficient of interaction or influence expresses by coefficients of inertia correction in the form:

$$a_{11\mathrm{int}\,eraction} = -\hat{a}_{01}\frac{\hat{\mu}_{01}\xi^3}{\Delta(\xi)}(\cos\xi - \mu_P\xi\sin\xi) \tag{31}$$

Then the coefficient of the mass inertia $a^*_{11} = a_{11} + \hat{a}_{11} + a_{11\mathrm{int}\,eraction}$ contains three parts: one coefficient of the mass inertia of the discrete subsystem, second part is the coefficient of mass inertia of dynamical constraint, and the third part is coefficient of the of interaction or influence continuous subsystem to the discrete subsystem.

Taking into consideration that for hybrid system with same subsystems as in this considered subsystems, but with statically connection by spring frequency equation is in the form (see References [2] and [3] by Hedrih):

$$\left| \overline{\mathbf{C}} - \xi^2 u_0 \overline{\mathbf{A}} + \left[1 - \frac{\kappa\xi}{\Delta(\xi)} (\cos\xi - \mu_P \xi \sin\xi) \right] \mathbf{I}_0 \right| = 0 \tag{32}$$

where: $\Delta(\xi) = \xi\{\mu_P \xi[(\mu_0 \xi^2 - \kappa)\sin\xi - \xi\cos\xi] - [(\mu_0\xi^2 - \kappa)\cos\xi + \xi\sin\xi]\}$ and $c_e = \frac{\mathbf{EA}}{\ell}$, $\kappa = \frac{c_0}{c_e}$, c_0 is the coefficient of the rigidity of the spring - static connection between subsystems. Then, we can see that this equation (32) consists of two parts: one part is expression of the frequency equations of the discrete system oscillations, and second of the deformable body frequency equation connected by one member with previous, which is correction of the first coefficient of the quasielastic matrix coefficient in the form:

$$\bar{c}^*_{11} = \bar{c}_{11} + 1 - \frac{\kappa\xi}{\Delta(\xi)} (\cos\xi - \mu_P \xi \sin\xi) \tag{33}$$

or in the form:

$$c^*_{11} = c_{11} + c_0 - c_0 \frac{\kappa\xi}{\Delta(\xi)} (\cos\xi - \mu_P \xi \sin\xi) \tag{33*}$$

From last expression (33) we can conclude that interaction between statically coupled subsystems is expressed by coefficient of interaction or influence is by the coefficient of quasielastic correction in the form:

$$c_{1\,1interaction} = -c_0 \frac{\kappa\xi}{\Delta(\xi)} (\cos\xi - \mu_P \xi \sin\xi) \tag{34}$$

then coefficient of the spring rigidity is $c^*_{11} = c_{11} + \hat{c}_{11} + c_{11interaction}$ it contains three parts: one is the quasyelastic coefficient of the discrete subsystem, second part is the coefficient of elasticity of the static constraint, and the third part is coefficient of the of interaction or influence continuous subsystem to the discrete subsystem.

3. THE SET OF THE FREQUENCY EQUATION THEOREMS OF SMALL OSCILLATIONS OF THE HYBRID SYSTEM CONTAINING COUPLED DISCRETE AND CONTINUOUS SUBSYSTEMS.

In the paper [2] and in this paper, we presented the research results of the analysis interactions between discrete subsystem and continuous subsystem to the frequency equations for the case of the static or dynamic constraints for realizations of the coupling. On the basis of these research results we can formulate the following theorems:

Theorem 1. Let us have two subsystems:

* one ideally elastic rod, as a continuous system solid deformable body with following parameters: **E**, ρ, ℓ , **A**, and with two rigid bodies on the ends with weights at the free ends with masses m_p and m_0 (see Figure 1); and

* second a linear subsystem of discrete material particles with n degrees of freedom and described by n generalized coordinates x_i, $i = 1,2,3,...,n$, and corresponding matrix **A** of inertia coefficients and matrix **C** quazielastic coefficients:

$$\begin{aligned} \mathbf{A} &= (a_{ij}), \ \ i = 1,2,3,...,n; \quad j = 1,2,3,...,n \\ \mathbf{C} &= (c_{ij}), \ \ i = 1,2,3,...,n; \quad j = 1,2,3,...,n \end{aligned} \tag{35}$$

* let these subsystems be coupled by dynamical constraint in the form of the rolling element (see Figure 1) with kinetic parameters m, $\mathbf{J}_C$, R between coordinates $u(\ell, t) = x_0(t)$ and x_1, then the frequency equation is in the following transcendent form:

$$\left| \overline{\mathbf{C}} - \xi^2 u_0 \left(\overline{\mathbf{A}} + \left[\tilde{\mu}_{11} - \tilde{\mu}_{01} \frac{\hat{\mu}_{01}\xi^3}{\Delta(\xi)} (\cos\xi - \mu_P \xi \sin\xi) \right] \mathbf{I}_0 \right) \right| = 0 \tag{36}$$

where $\Delta(\xi) = \xi^2 \{\mu_P \xi[\xi(\mu_0 + \hat{\mu}_{00})\sin\xi - \cos\xi] - [\xi(\mu_0 + \hat{\mu}_{00})\cos\xi + \sin\xi]\}$ and notations used in the form (16).

Theorem 2. Let us have two subsystems:

* one ideally elastic rod, as a continuous system solid deformable body with following parameters: **E**, ρ, ℓ , **A**, and with two rigid bodies on the ends with weights at the free ends with masses m_p and m_0 (see Figure 1); and

* second a linear subsystem of discrete material particles with n degrees of freedom and described by n generalized coordinates x_i, $i = 1,2,3,...,n$, and corresponding matrix **A** of inertia coefficients and matrix **C** quazielastic coefficients:

$$\begin{aligned} \mathbf{A} &= (a_{ij}), \ \ i = 1,2,3,...,n; \quad j = 1,2,3,...,n \\ \mathbf{C} &= (c_{ij}), \ \ i = 1,2,3,...,n; \quad j = 1,2,3,...,n \end{aligned} \tag{37}$$

* let these subsystems be coupled by dynamical constraint in the form of a rolling element (see Figure 1) with kinetic parameters m, $\mathbf{J}_C$, R between coordinates $u(\ell, t) = x_0(t)$ and x_1, then the frequency equation is in the form of frequency equation of the discrete subsystem

$$\left| \overline{\mathbf{C}} - \xi^2 u_0 \overline{\mathbf{A}}^* \right| = 0 \tag{38}$$

in which coefficient of inertia a_{11} is extended by two terms

$$a^*_{11} = a_{11} + \left[\hat{a}_{11} - \hat{a}_{01} \frac{\hat{\mu}_{01}\xi^3}{\Delta(\xi)} (\cos\xi - \mu_P \xi \sin\xi) \right] \tag{39}$$

and that interaction between subsystems dynamically coupled is expressed by coefficient of interaction or influence expressed by coefficient of inertia correction in the form:

$$a_{11\,interaction} = -\hat{a}_{01} \frac{\hat{\mu}_{01}\xi^3}{\Delta(\xi)} (\cos\xi - \mu_P \xi \sin\xi) \tag{40}$$

where $\Delta(\xi)=\xi^2\{\mu_P\xi[\xi(\mu_0+\hat{\mu}_{00})\sin\xi-\cos\xi]-[\xi(\mu_0+\hat{\mu}_{00})\cos\xi+\sin\xi]\}$ and notations used in the form (16). The coefficient of the mass inertia $a^*_{11}=a_{11}+\hat{a}_{11}+a_{11\,\mathrm{int}\,eraction}$ contains three parts: one is the coefficient of the mass inertia of discrete subsystem, second part is the coefficient of the mass inertia of dynamical constraint, and the third part is the coefficient of interaction or influence of continuous subsystem on the discrete subsystem.

Theorem 3. Let us have two subsystems:

* one ideally elastic rod, as a continuous system solid deformable body with following parameters: **E**, ρ, ℓ , **A**, and with two rigid bodies on the ends with weights at the free ends with masses m_p and m_0 (see Figure 1.); and

* second a linear subsystem of discrete material particles with n degrees of freedom and described by n generalized coordinates x_i, i = 1,2,3,...,n, and corresponding matrix **A** of inertia coefficients and matrix **C** quazielastic coefficients:

$$\begin{aligned}\mathbf{A}&=(a_{ij}),\ i=1,2,3,...,n;\quad j=1,2,3,...,n\\ \mathbf{C}&=(c_{ij}),\ i=1,2,3,...,n;\quad j=1,2,3,...,n\end{aligned}\tag{41}$$

* let these subsystems be coupled by static constraint in the form of spring element of negligible mass and with coefficient of rigidity c_0 between elements with coordinates $u(\ell,t)=x_0(t)$ and x_1, then the frequency equation is in the following transcendent form:

$$\left|\overline{\mathbf{C}}-\xi^2u_0\overline{\mathbf{A}}+\left[1-\frac{\kappa\xi}{\Delta(\xi)}(\cos\xi-\mu_P\xi\sin\xi)\right]\mathbf{I}_0\right|=0\tag{42}$$

where: $\Delta(\xi)=\xi\{\mu_P\xi[(\mu_0\xi^2-\kappa)\sin\xi-\xi\cos\xi]-[(\mu_0\xi^2-\kappa)\cos\xi+\xi\sin\xi]\}$ and notations used in the form (16).

Theorem 4. Let us have two subsystems:

* one ideally elastic rod, as a continuous system solid deformable body with following parameters: **E**, ρ, ℓ , **A**, and with two rigid bodies on the ends with weights at the free ends with masses m_p and m_0 (see Figure 1.); and

* second a linear subsystem of discrete material particles with n degrees of freedom and described by n generalized coordinates x_i, i = 1,2,3,...,n, and corresponding matrix **A** of inertia coefficients and matrix **C** quazielastic coefficients:

$$\begin{aligned}\mathbf{A}&=(a_{ij}),\ i=1,2,3,...,n;\quad j=1,2,3,...,n\\ \mathbf{C}&=(c_{ij}),\ i=1,2,3,...,n;\quad j=1,2,3,...,n\end{aligned}\tag{43}$$

* let these subsystems be coupled by static constraint in the form of a spring element of negligible mass and with coefficient of rigidity c_0 between elements with coordinates $u(\ell,t)=x_0(t)$ and x_1, then the frequency equation is in the form of the frequency equation of the discrete subsystem

$$\left|\overline{\mathbf{C}}^*-\xi^2u_0\overline{\mathbf{A}}\right|=0\tag{44}$$

in which element of the quasielastic matrix c_{11} is extended by two additional parts:

$$c^*_{11} = c_{11} + c_0 - c_0 \frac{\kappa\xi}{\Delta(\xi)}(\cos\xi - \mu_P \xi \sin\xi) \tag{45}$$

where: $\Delta(\xi) = \xi\{\mu_P \xi[(\mu_0 \xi^2 - \kappa)\sin\xi - \xi\cos\xi] - [(\mu_0\xi^2 - \kappa)\cos\xi + \xi\sin\xi]\}$ and notations used in the form (16). The interaction between subsystems statically coupled is expressed by coefficient of interaction or influence expressed by coefficient of quasielastic correction in the form:

$$c_{1\,1interaction} = -c_0 \frac{\kappa\xi}{\Delta(\xi)}(\cos\xi - \mu_P \xi \sin\xi) \tag{46}$$

then coefficient of the spring rigidity is $c^*_{11} = c_{11} + \hat{c}_{11} + c_{11interaction}$ *and* contains three parts: one coefficient of the quasielastic coefficient of the discrete subsystem, second part coefficient of elasticity of static constraint, and third part is coefficient of interaction or influence of continuous subsystem on the discrete subsystem.

4. Special cases of the frequency equation of the dynamically coupled longitudinal oscillations of the elastic rod and discrete subsystem of the material particles

In the developed form derived frequency equation (29) of free oscillations of the hybrid system of the considered dynamically coupled subsystems take the following form:

$$\begin{vmatrix} k_{11} - \xi^2 u_0\left(\tilde{\mu}_{11} + \mu_{11} - \tilde{\mu}_{01}\frac{\hat{\mu}_{01}\xi^3}{\Delta(\xi)}(\cos\xi - \mu_P\xi\sin\xi)\right) & k_{12} - \xi^2 u_0\mu_{12} & k_{13} - \xi^2 u_0\mu_{13} & k_{1n} - \xi^2 u_0\mu_{1n} \\ k_{21} - \xi^2 u_0\mu_{21} & k_{22} - \xi^2 u_0\mu_{22} & k_{23} - \xi^2 u_0\mu_{23} & k_{2n} - \xi^2 u_0\mu_{2n} \\ k_{31} - \xi^2 u_0\mu_{31} & k_{32} - \xi^2 u_0\mu_{32} & k_{33} - \xi^2 u_0\mu_{33} & k_{3n} - \xi^2 u_0\mu_{3n} \\ & & & \\ k_{n1} - \xi^2 u_0\mu_{n1} & k_{2n} - \xi^2 u_0\mu_{2n} & k_{3n} - \xi^2 u_0\mu_{3n} & k_{nn} - \xi^2 u_0\mu_{nn1} \end{vmatrix} = 0 \tag{47}$$

For the case of the coupled elastic rod longitudinal oscillations and chain discrete material particles system oscillations, the previous frequency equations take the following form:

$$\begin{vmatrix} k_1 - \xi^2 u_0\left(\tilde{\mu}_{11} + \mu_{11} - \tilde{\mu}_{01}\frac{\hat{\mu}_{01}\xi^3}{\Delta(\xi)}(\cos\xi - \mu_P\xi\sin\xi)\right) & -k_1 & & & \\ -k_1 & k_1 + k_2 - \xi^2 u_0\mu_2 & -k_2 & & \\ & -k_2 & k_2 + k_3 - \xi^2 u_0\mu_3 & \vdots & \\ & & \cdots & \cdots & \\ & & & & -k_{n-1} \\ & & & \vdots & k_{n-1} + k_n - \xi^2 u_0\mu_n \end{vmatrix} = 0 \tag{48}$$

For the case when elastic rod is dynamically connected with one material particle with one spring we obtain:

$$k_1-\xi^2 u_0\left(\widetilde{\mu}_{11}+\mu_{11}-\widetilde{\mu}_{01}\frac{\hat{\mu}_{01}\xi^3}{\Delta(\xi)}(\cos\xi-\mu_P\xi\sin\xi)\right)=0 \tag{49}$$

and taking into account expression (20) we can write the following :

$$tg\xi=\frac{u_0\widetilde{\mu}_{01}\hat{\mu}_{01}\xi^3+\xi[k_1-\xi^2u_0(\widetilde{\mu}_{11}+\mu_{11})]\{\mu_P+\mu_0+\hat{\mu}_{00}\}}{\{[k_1-\xi^2u_0(\widetilde{\mu}_{11}+\mu_{11})][\mu_P(\mu_0+\hat{\mu}_{00})\xi^2-1]+\mu_P u_0\widetilde{\mu}_{01}\hat{\mu}_{01}\xi^4\}} \tag{50}$$

For the free left end of the rod $\mu_P\to 0$

$$tg\xi=-\xi(\mu_0+\hat{\mu}_{00})-\frac{u_0\widetilde{\mu}_{01}\hat{\mu}_{01}\xi^3}{[k_1-\xi^2u_0(\widetilde{\mu}_{11}+\mu_{11})]} \tag{51}$$

For the case when one end of the rod is fixed – case of the cantilever rod, in the previous frequency equation we can introduce $\mu_P\to\infty$, and than we obtain:

$$tg\xi=\frac{[k_1-\xi^2u_0(\widetilde{\mu}_{11}+\mu_{11})]}{\xi(\mu_0+\hat{\mu}_{00})[k_1-\xi^2u_0(\widetilde{\mu}_{11}+\mu_{11})]+u_0\widetilde{\mu}_{01}\hat{\mu}_{01}\xi^3} \tag{52}$$

For the case of a free material particle connected by one spring to the rod we can write:

$$\xi tg\xi=\frac{(1-\xi^2u_0\mu_1)}{(\mu_0+\kappa u_0\mu_1-\mu_0\xi^2u_0\mu_1)} \tag{53}$$

For two material particles connected to a rod as a chain, we obtain:

$$\left\{k_1-\xi^2 u_0\left(\widetilde{\mu}_{11}+\mu_{11}-\widetilde{\mu}_{01}\frac{\hat{\mu}_{01}\xi^3}{\Delta(\xi)}(\cos\xi-\mu_P\xi\sin\xi)\right)\right\}(k_1+k_2-\xi^2u_0\mu_2)-k_1^2=0 \tag{54}$$

where:

$$\Delta(\xi)=\xi^2\{\mu_P\xi[\xi(\mu_0+\hat{\mu}_{00})\sin\xi-\cos\xi]-[\xi(\mu_0+\hat{\mu}_{00})\cos\xi+\sin\xi]\} \tag{55}$$

For three material particles chain the frequency equation is:

$$\begin{vmatrix} k_1-\xi^2 u_0\left(\widetilde{\mu}_{11}+\mu_{11}-\widetilde{\mu}_{01}\frac{\hat{\mu}_{01}\xi^3}{\Delta(\xi)}(\cos\xi-\mu_P\xi\sin\xi)\right) & -k_1 & \\ -k_1 & k_1+k_2-\xi^2u_0\mu_2 & -k_2 \\ & -k_2 & k_2+k_3-\xi^2u_0\mu_3 \end{vmatrix}=0 \tag{56}$$

For the case when we have discrete material particles homogeneous chain, frequency equation obtains the following form:

$$\begin{vmatrix} 1-\xi^2\tilde{u}_0\left(\tilde{\mu}_{11}+1-\tilde{\mu}_{01}\frac{\hat{\mu}_{01}\xi^3}{\Delta(\xi)}(\cos\xi-\mu_P\xi\sin\xi)\right) & -1 & & & & \\ -1 & 2-\xi^2\tilde{u}_0 & -1 & & & \\ & -1 & 2-\xi^2\tilde{u}_0 & \vdots & & \\ & & \dots & \dots & & \\ & & & & 2-\xi^2\tilde{u}_0 & -1 \\ & & & & -1: & 2-\xi^2\tilde{u}_0 \end{vmatrix}=0 \quad (57)$$

For the special case of the three material particles homogeneous chain, the frequency equation obtains the following form:

$$[(2-\xi^2\tilde{u}_0)^2-1]\left\{1-\xi^2\tilde{u}_0\left(\tilde{\mu}_{11}+1-\tilde{\mu}_{01}\frac{\hat{\mu}_{01}\xi^3}{\Delta(\xi)}(\cos\xi-\mu_P\xi\sin\xi)\right)\right\}-(2-\xi^2\tilde{u})=0 \quad (58)$$

5. ANALOGY BETWEEN THE FREQUENCY EQUATION OF THE COUPLED LONGITUDINAL OSCILLATIONS OF THE ELASTIC ROD AND DISCRETE SYSTEM OF THE MATERIAL PARTICLES AND COUPLED TORSION OSCILLATIONS OF THE ELASTIC ROD AND CORRESPONDING DISCRETE SYSTEM OF THE MATERIAL PARTICLES

By using analogy (see Ref. [28] and [29]) between two systems, and especially between longitudinal and torsional oscillations of elastic rod with circle cross section we can use previous analytical results for determining the frequency equation of coupled small oscillations of mixed systems presented in Figure 1. and Figure 2.

In general case hybrid system consists of two subsystems: one elastic rod-shaft, whose axis is straight, with parameters: $\mathbf{G}$, ρ, ℓ, $\mathbf{A}$, $\mathbf{I}_0$, and with two rigid discs at the free ends with mass inertia moment with respect to the shaft axis: $\mathbf{J}_P$ and $\mathbf{J}_0$. This rod-shaft is constrained by torsion spring with rigidity c_0 and coupled with discrete systems with n degrees of freedom. For example, this discrete subsystem is mechanism in the form of chain system of n material particles (or rigid bodies) with generalized masses m_i, $i = 1,2,3,...,n$, torsion (rotation) movable along circle line coaxial to the rod-shaft's axis; these masses are connected by torsion springs with rigidities c_i, $i = 1,2,3,...,n$. We consider connections between the torsional vibrations of elastic rod-shaft and free oscillations of generalized chain material particles system-mechanism (examples in Figure 2). Let us determine the frequency equations of the defined mixed system of the coupled discrete subsystem of rigid bodies and continuous subsystem.

Using the analogy

$$\theta(x,t)\Rightarrow u(x,t) \qquad \mathbf{J}_P\Rightarrow m_P \qquad \mathbf{J}_P\Rightarrow m_P \qquad x_i\Rightarrow\theta_i$$

and taking into account that notation of the kinetic parameters for the case of torsional vibrations we have:

$$\mu_P=\frac{\mathbf{J}_P}{\rho\mathbf{I}_0\ell} \qquad \mu_o=\frac{\mathbf{J}_0}{\rho\mathbf{I}_0\ell} \qquad \xi=\lambda\ell$$

$$\omega^2=\lambda^2\frac{\mathbf{G}}{\rho}=\frac{\xi^2}{\ell^2}\frac{\mathbf{G}}{\rho}=\xi^2\omega_0^2 \qquad \omega_0^2=\frac{\mathbf{G}}{\rho\ell^2}$$

$$c_t = \frac{\mathbf{GI}_0}{\ell}, \qquad \kappa\frac{c_0}{c_t}, \qquad u_0 = \frac{m_0\omega_0^2}{c_0}.$$

$$\hat{\mu}_{00} = \frac{\hat{a}_{00}}{\rho\mathbf{I_0}\ell} \quad \hat{\mu}_{01} = \frac{\hat{a}_{01}}{\rho\mathbf{I_0}\ell} \quad \tilde{\mu}_{01} = \frac{\hat{a}_{01}}{\mathbf{J}_0} \quad \tilde{\mu}_{11} = \frac{\hat{a}_{11}}{\mathbf{J}_0} \tag{59}$$

And those are analogous to the notation (16). For the kinetic parameters in the case when the system is with longitudinal vibrations, all derived frequency equations are valid when taking into account analogous notations (16) and (59).

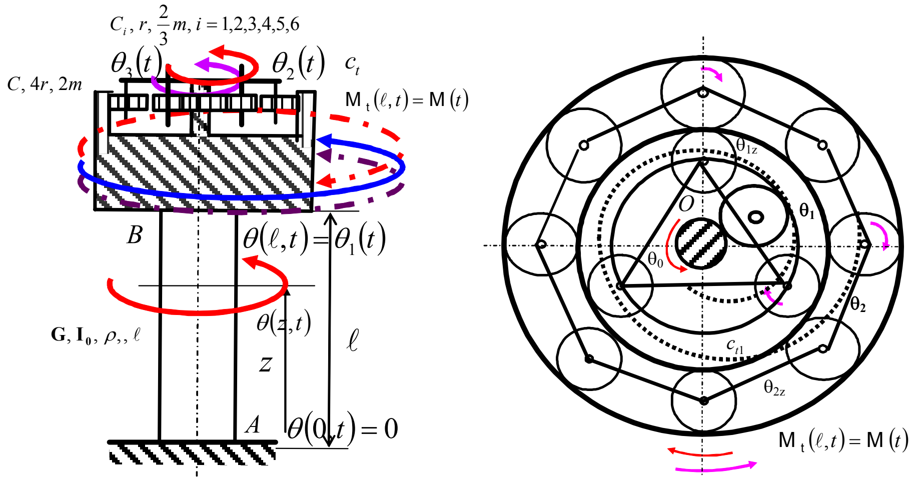

Fig. 2. Small oscillations of the hybrid system of the coupled discrete and continuous subsystems - Torsion oscillations of the cantilever shaft dynamical coupled with multi body mechanisms with two chains of the changeable numbers of discs.

6. Concluding Remarks

From the obtained analytical results for natural longitudinal vibrations of an elastic rod dynamically coupled with material particle discrete system, it can be seen that connections are convenient for changing characteristic function depending on discrete system material parameters, and that fundamental eigen-function depending on space coordinate is dependent on boundary conditions and kinetic properties of dynamical constraint and discrete subsystem coupled into hybrid system.

In this paper we returned to classical problems of the theory of oscillations, dynamically coupled elastic body and subsystem of discrete material particles on selected examples and at the same time we determined the corresponding frequency equations.

Results of a qualitative analysis on frequency function show that this equation contains the members that express the influence of continual subsystem to the discrete subsystem on frequency equations through mass inertia coefficient and that member is expressed the influence of deformable body through transcendent member. We can see from this inertia coefficient the perturbations of frequency spectre of own circular *frequencies of discrete subsystem, by deformable body oscillations or vice versa. Similar disturbances can be seen on the frequency spectre of a discrete system but with opposite effects. We can see "the continualization of frequency spectra of discrete subsystem" on the roots of the frequency equation. At the same time we can interpret these results as discretization of the part of frequency spectra of continual system as a result of coupling with a discrete system. It should also be stated here an analogy used between these hybrid systems with dynamically, and also statically coupled subsystems, continuous and discrete when it is possible to establish a direct analogy between longitudinal and torsional oscillations of deformable body with circular annular cross-sections. That enabled an analytical analysis to be conducted for one type of hybrid systems and results to be used on another type. And at the end, it should be stressed again that the goal of this paper was the solution of a classical but very concurrent task since the literature contains a very small number of examples of such a task. Methodology of continuum discretization and of continualization of discrete system which meet at border cases of study of properties of real systems is very useful.*

At end of this concluding remarks, we can define one generalized frequency equation theorem of small oscillations of the hybrid system containing coupled discrete and continuous subsystems.

Generalized Theorem 5. Let us have two subsystems:

* one ideally elastic rod, as a continuous system solid deformable body with following parameters: **E**, ρ, ℓ , **A**, and with two rigid bodies on the ends with weights at the free ends with masses m_p and m_0 (see Figure 1.); and

* second a linear subsystems of discrete material particles with n degrees of freedom and described by n generalized coordinates x_i, $i = 1,2,3,...,n$, and corresponding matrix **A** of inertia coefficients and matrix **C** quazielastic coefficients:

$$\begin{aligned} \mathbf{A} &= (a_{ij}),\ i = 1,2,3,...,n; \quad j = 1,2,3,...,n \\ \mathbf{C} &= (c_{ij}),\ i = 1,2,3,...,n; \quad j = 1,2,3,...,n \end{aligned} \tag{60}$$

* let these subsystems be coupled by a dynamical constraint in the form of rolling element (see Figure 1) with kinetic parameters m, $\mathbf{J}_C$, R between coordinates $u(\ell,t) = x_0(t)$ and x_1, and also by a static constraint in the form of spring element of negligible mass and with coefficient of rigidity c_0 between elements with coordinates $u(\ell,t) = x_0(t)$ and x_1, then the frequency equation is in the following transcendent form:

$$\left| \overline{\mathbf{C}} + \left[1 - \frac{\kappa\xi}{\Delta_{st}(\xi)}(\cos\xi - \mu_P \xi \sin\xi)\right]\mathbf{I}_0 - \xi^2 u_0 \left(\overline{\mathbf{A}} + \left[\tilde{\mu}_{11} - \tilde{\mu}_{01}\frac{\hat{\mu}_{01}\xi^3}{\Delta_{din}(\xi)}(\cos\xi - \mu_P\xi\sin\xi)\right]\mathbf{I}_0\right)\right| = 0 \tag{61}$$

where

$$\Delta_{st}(\xi) = \xi\{\mu_P\xi[(\mu_0\xi^2 - \kappa)\sin\xi - \xi\cos\xi] - [(\mu_0\xi^2 - \kappa)\cos\xi + \xi\sin\xi]\} \tag{62}$$

$$\Delta_{din}(\xi) = \xi^2\{\mu_P\xi[\xi(\mu_0 + \hat{\mu}_{00})\sin\xi - \cos\xi] - [\xi(\mu_0 + \hat{\mu}_{00})\cos\xi + \sin\xi]\} \tag{63}$$

and notations used in the form (16).

If we have hybrid linear system containing many continuous linear subsystems and discrete linear subsystem with n degrees of freedom it is possible to make "corrections" of mass inertia coefficients and quazielastic coefficients in the corresponding matrix and compose frequency equation in the form $\left|\overline{\mathbf{C}}^{*}-\xi^{2}u_{0}\overline{\mathbf{A}}^{*}\right|=0$ but result is transcendent characteristic equation with infinite number of the roots – characretistic eigen numbers, as well as infinite numbers of eigen frequencies.

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TEOREME O FREKVENTNIM JEDNAČINA MALIH OSCILACIJA HIBRIDNIH SISTEMA, KOJI SADRŽE SPREGNUTE DISKRETNE I KONTINUALNE PODSISTEME

Katica (Stevanović) Hedrih

Koristeći primer hibridnog sistema sastavljenog od dinamički spregnutih podsistema krutih tela i kontinualnog podsistema, postavljena je metoda dobijanja frekventnih jednačina malih oscilacija hibridnih sistema. Takodje je postavljeno nekoliko teorema koje definišu osobine frekventnih jednačina posmatrane klase hibridnih sistema. Koristeći primere i analogiju izmedju nekih klasa hibridnih sistema pokazana je analogija izmedju njihovih frekventnih jednačina. Specijalni slučajevi diskretizacije i kontinualizacije spregnutih podsistema u odnosu na njihove skupove sopstvenih kružnih frekvencija i frekventnih jednačina malih oscilacija su proučeni. Izveden je i dokazan skup teorema o frekventnim jednačinama je malih oscilacija hibridnih sistema koji se sadrže, dinamičkim vezama spregnute diskretne i kontinualne podsisteme.

Ključne reči: *Hibridni system, spregnuti podsistemi, kruta tela, deformabilna tela, frekventne jednačine, teoreme.*